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Mixed and hybrid finite element methods for convection-diffusion equations and their relationships with finite volume

M. Fortin[1], **A. Serghini Mounim**[2]

[1] Département de Mathématiques et Statistique, Université Laval, Québec, G1K 7P4, Canada
e-mail: mfortin@giref.ulaval.ca

[2] Département de Mathématiques et Statistique, Université de Moncton, Moncton, N.-B., E1A 3E9, Canada
e-mail: serghia@umoncton.ca

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Abstract. We study the relationship between finite volume and mixed finite element methods for the the hyperbolic conservation laws, and the closely related convection-diffusion equations. A general framework is proposed for the derivation and a functional framework is developed which could allow the analysis of relating finite volume (FV) schemes. We show via two non-standard formulations, that numerous FV schemes, including centred, upwind, Lax-Friedrichs, Roe, Engquist-Osher, the central Nessyahu-Tadmor schemes, etc., can be recovered in the unique dual mixed and hybrid (DMH) finite element framework. That makes possible a better understanding of these FV schemes. Moreover, the large number of DMH finite element results available can then give the analysis of these FV methods in a unified fashion. Furthermore, stabilized methods are proposed. In particular, interpretation in terms of the Lagrange multiplier of flux-limiter is given.

We end by presenting numerical results to validate the newly proposed stabilized schemes.

1 Introduction

Mixed formulations for performing finite element approximations of partial differential equations are appealing from a theoretical point of view, since a standard framework and large number of results are available for carrying out their analysis.

In other respects, mixed finite element methods have been proved effective for a number of engineering problems. They provide good and efficient approximations to stress variables, have the capability of dealing with rough coefficients, and are the natural choice for equations coupling velocity and pressure or stress and displacement variables. Nevertheless, a popular and more appropriate method for discretization of conservation laws is still the FV method, used extensively in computational fluid dynamics (CFD), based on piecewise constant approximation of the solution. The FV method is widely used in applications because of its great ability in handling the convective terms in particular; it also insures the local conservation of physical quantities. However, the FV method has problems with the approximation of diffusion terms; the analysis and the extension to the multidimensional setting are not standardized, and these schemes are sensitive to the triangulation of the domain (convergence problems on elements with arbitrarily large aspect ratios, so-called anisotropic finite element meshes).

Furthermore, for the diffusion problem, the relationship between mixed finite element methods and finite volume was established among others by Farhloul and Fortin [23] (rectangular case), and Baranger et al. [7] (triangular case). This paper extends this connection, presenting a general framework for the derivation and a functional framework for the analysis of finite volume schemes, applied to the discretization of general conservation laws. The procedure is based on a dual mixed and hybrid (DMH) finite element formulation, and its connection with the FV method for nonlinear hyperbolic conservation laws,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1}$$

and the closely related convection-diffusion equations,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

with given data $u(x,0) = u_0(x)$ and corresponding suitable boundary conditions. Here $u := u(x,t)$ is a conserved quantity, $f(u)$ is a convective flux, and $\frac{\partial u}{\partial x}$ is a dissipation flux.

These equations are of great practical importance since they arise in fluid flows, reactive flows, groundwater flows, non-Newtonian flows, traffic flows, two-phase flows in oil reservoirs, etc. They also govern a variety of physical phenomena that appear in aeronautics, astrophysics, meteorology, semiconductors, financial modelling, front propagation, and other areas.

The numerical solution of the advective-diffusive transport equation is a problem of great importance because many problems in science and engineering involve such mathematical models. When the process is advection dominated, the problem is especially difficult.

On the other hand, the discrete conservation of a numerical algorithm for (1) or (2) is important in order to keep the correct location of the discontinuities. Hou and LeFloch in [36] have shown that, if a nonconservative scheme for (1.1) converges, it converges to a solution of $\partial_t u + \partial_x f(u) = \mu$, where μ is a Borel measure source term that is expected to be zero in the region where the solution u is smooth and concentrated where u is not smooth. Then the schemes insuring the local conservation are in order.

To achieve this, a mixed method is considered for this work because it conserves mass on a cell-by-cell basis. Next, the convective and diffusion fluxes are introduced as auxiliary variables, and Lagrange multipliers are used to impose interelement continuity.

In this first paper, we start with the one-dimensional problem and extract the principal properties of the approach, and we explain how some of the classical volume schemes can be derived in a simple and systematic manner. The extension of the methods to multidimensional hyperbolic and the related convection-diffusion equations, and also to the Navier-Stokes equations, will be the subject of other papers.

This paper is organized as follows. We start in the next section with the one-dimensional problem. In Sect. 2.1, first, we introduce the convective and diffusion fluxes as auxiliary variables in each cell. Second, we relax the continuity of the fluxes across the interelement via two Lagrange multipliers, to obtain the abstract dual mixed and hybrid (DMH1) formulation. In Sect. 2.2, we extend the above approach; this yields the DMH2 formulation. The finite element discretization of the two methods and a few remarks are given in Sect. 3. In Sect. 4, we show how to obtain finite volume schemes from DMH finite element methods. Stabilized methods are reported in Sect. 5 because we are concerned with the numerical approximation of the solutions of convection-diffusion problems in which the convection or transport dominates the diffusion. The first combines the upwind feature without the complicated and costly resolution of Riemann problems. The last utilizes the limiter-flux strategy.

In Sect.6, we discuss how we can establish the extension of the (DMH2) method to both convection-diffusion equations and systems of equations. We end by presenting numerical results to validate the newly proposed schemes.

2 Dual mixed and hybrid finite element methods

2.1 Formulation 1

We begin this section by considering the simple one-dimensional model to extract the fundamental points.

Throughout the paper, for a bounded interval $I = (a, b)$ of $\mathbb{R}$, we set $I_T = I \times (0, T]$, with $T > 0$ denoting the final time; let $L^2(I)$ and

$H^1(I)\,(H^1(I) \subset C(I))$ be the usual Lebesgue space and Sobolev space, respectively.

We next consider the initial-boundary value problem:
find $u : I_T \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial (f(u))}{\partial x} = 0 & \text{in} \quad I_T, \\ u(x,0) = u_0(x) & \text{in} \quad I, \end{cases} \tag{3}$$

with corresponding suitable boundary conditions.

If we introduce the dissipation and the convective fluxes as new unknowns, then problem (3) can be formulated as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial p}{\partial x} + \dfrac{\partial \hat{p}}{\partial x} = 0 & \text{in} \quad I_T, \\ p = \dfrac{\partial u}{\partial x} & \text{in} \quad I, \\ \hat{p} = f(u) & \text{in} \quad I, \end{cases}$$

with given initial data $u(x,0) = u_0(x)$, and corresponding suitable boundary conditions.

In order to describe our formulation of the above problem, we start by introducing more notation. Consider an arbitrary partition of I into N subintervals $I_i = \{x \in I \mid x_i < x < x_{i+1}\}$ of length $h_i := x_{i+1} - x_i$ with $i = 0, 1, \ldots, N$. We also consider a time step Δt and define the times $t^n = n\Delta t$, for $n = 1, 2, \ldots, M$; by Δt, we mean $\Delta t = \dfrac{T}{M}$. Moreover, we also need the spaces:

$$\begin{aligned} M_2 &= \left\{ \mu : \{x_0, x_1, \ldots, x_{N+1}\} \to \mathbb{R} \right\}, \\ \overline{M}_2 &= \left\{ \mu \in M_2 \mid \mu_0 = \mu_{N+1} = 0 \right\}, \\ M_1 &= L^2(I) \end{aligned}$$

and

$$X = \left\{ q \in L^2(I), q|_{I_i} \in H^1(I_i) \right\}.$$

The hybridization of the mixed formulation can be obtained by relaxing the continuity of p and $\hat{p}$ across interelement boundaries, with Lagrange multipliers λ and $\hat{\lambda}$ respectively:

$$\begin{aligned} \sum_i \int_{I_i} pq\,dx = &-\sum_i \int_{I_i} \overline{u} \frac{dq}{dx}\,dx \\ &+ \sum_i \left(\lambda_{i+1} q(x_{i+1}^-) - \lambda_i q(x_i^+)\right) \quad \forall q \in X, \end{aligned}$$

and

$$\sum_i \int_{I_i} \hat{p}\, q\, dx = \sum_i \int_{I_i} \overline{f}(u)\, q\, dx + \sum_i \left(\hat{\lambda}_{i+1}\, q(x_{i+1}^-) - \hat{\lambda}_i q(x_i^+)\right) \quad \forall q \in X,$$

for $0 \le i \le N$, and where

$$\overline{u} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} u\, dt, \quad \overline{f}(u) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u)\, dt,$$

and $q(x_i^\mp)$ are the left and right traces of q at x_i $\left(q(x_i^\pm) := \lim_{\epsilon \longrightarrow 0} q(x_i \pm \epsilon)\right)$. We then obtain the conservation of diffusion and convective fluxes in each cell. This completes the construction of our formulation. Moreover, note that λ could be interpreted as a trace of u in each subinterval I_i of I. Further, we will demonstrate the capital importance of the Lagrange multiplier $\hat{\lambda}$ in the numerical scheme.

Finally, let $u^n := u(., t^n)$; after the time discretization, the DMH1 formulation for (3) reads as follows.

For each $n = 0, 1, \ldots, M-1$, find $\left(u^{n+1}, (\lambda, \hat{\lambda})\right) \in M_1 \times \overline{M}_2^2$, and $(p, \hat{p}) \in X^2$ such that:

$$\left\{ \begin{aligned} & \sum_i \int_{I_i} \frac{u^{n+1} - u^n}{\Delta t} v\, dx - \nu \textstyle\sum_i \int_{I_i} \frac{dp}{dx} v\, dx \\ & \qquad\qquad + \sum_i \int_{I_i} \frac{d\hat{p}}{dx} v\, dx = 0, \quad \forall v \in M_1 \\ & \sum_i \int_{I_i} pq\, dx = -\sum_i \int_{I_i} \overline{u} \frac{dq}{dx}\, dx + \sum_i \left(\lambda_{i+1} q(x_{i+1}^-) \right.\\ & \qquad\qquad \left. - \lambda_i q(x_i^+)\right) \quad \forall q \in X, \\ & \sum_i \int_{I_i} \hat{p}\, q\, dx = \sum_i \int_{I_i} \overline{f}(u)\, q\, dx + \sum_i \left(\hat{\lambda}_{i+1}\, q(x_{i+1}^-) \right.\\ & \qquad\qquad \left. - \hat{\lambda}_i q(x_i^+)\right) \quad \forall q \in X, \\ & \sum_i (p(x_i^+) - p(x_i^-))\, \mu = 0 \quad \forall \mu \in \overline{M}_2, \\ & \sum_i (\hat{p}(x_i^+) - \hat{p}(x_i^-))\, \mu = 0 \quad \forall \mu \in \overline{M}_2, \end{aligned} \right. \tag{4}$$

for $0 \le i \le N$, with given data $u(x, 0) = u_0(x)$ and corresponding suitable boundary conditions, $\lambda_i = \lambda(x_i)$ and $\hat{\lambda}_i = \hat{\lambda}(x_i)$.

This may look like a complicated way of discritizing a simple problem. We shall see that it enables us to interpret some standard schemes, in particular flux limiter schemes, in terms of mixed methods.

2.2 Formulation 2

In order to extend the above approach, and to introduce the fundamental concepts of the proposed methodology, we restrain ourselves to transport equations. In Sect. 6, we discuss the extension of this approach to convection-diffusion equations and systems of equations. With this aim, we first introduce the hyperbolic conservation law with convex flux:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial\big(f(u)\big)}{\partial x} = 0 \quad \forall x \in \mathbb{R} \times (0, +\infty), \\ u_0(x) = u(x, 0) \quad \forall x \in \mathbb{R}. \end{cases} \tag{5}$$

Problem (5) may have discontinuous solutions (shocks) depending on the initial data u_0. Consider functions defined in a region Ω (space-time space), which may have jump discontinuities across an internal boundary Σ. In the finite element methods, Σ could be the union of all the inter-element boundaries in the space-time space ($\Sigma = \cup\Sigma_i$). If we can determine the trajectory Σ of a shock $x(t)$, then the shock solution is defined as

$$u(x,t) = \begin{cases} u^L(x,t) & \text{if} \quad x < x(t), \\ u^R(x,t) & \text{if} \quad x > x(t). \end{cases}$$

Shock solutions can be defined as weak solutions in the sense of distributions (Smoller [66]). It can be shown that the shock $x(t)$ must satisfy the Rankine-Hugoniot jump condition (RH),

$$-\frac{\nu_x}{\nu_t}[u] = [f(u)]. \tag{6}$$

Here ν is normal to the shock, it has components $\nu_x = -\frac{dx}{dt}$ (for convenience ν_x is oriented from left to right), $\nu_t = 1$, and $s = \frac{dx}{dt}$ is the shock speed. The square brackets stand for the jumps across Σ, $[u] = u\big(x(t)^+, t\big) - u\big(x(t)^-, t\big)$ $\big([f(u)] = f(u(x(t)^+, t)) - f(u(x(t)^-, t))\big)$, i.e., the limit on the positive side minus the limit on the negative side.

First, we start by assuming in our approach that the solution has discontinuities at each boundary of every I_i. Therefore, we have, in the usual sense in $\Omega \setminus (\cup\Sigma_i)$, the equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial(\hat{p})}{\partial x} = 0 \quad \forall x \in I_i \times (0, +\infty), \\ u_0(x) = u(x, 0) \quad \forall x \in I_i, \end{cases}$$

where we again set $\hat{p} = f(u)$ in each I_i. Next, on Σ_i, we have the jump formula (6).

Then, we suggest that the jump condition should be imposed on the numerical flux through element interfaces. This follows from the fact that, in the special case where $u(x_i^-, t)$ and $u(x_i^+, t)$ are connected by a single shock wave or contact discontinuity, the RH condition is satisfied for $u(x_i^-, t)$ and $u(x_i^+, t)$ for some speed s (the speed of the shock or contact).

As a second ingredient in the construction, this jump condition is enforced in a weak form by a Lagrange multiplier technique. In fact, if we restrict to right-going propagation, and for a given approximation of the local propagation speeds a_i ($a_i \geq 0$) at the cell boundaries x_i (for left-going propagation we take $a_i < 0$), we have

$$\sum_i \left(\hat{p}(x_i^+) - \hat{p}(x_i^-)\right)\mu = \sum_i a_i\,(\overline{u}_i^+ - \overline{u}_i^-)\,\mu \quad \forall\,\mu \in \overline{M}_2. \tag{7}$$

We again denote by $\hat{p}(x_i^{\mp})$ (resp. $u_i^{\mp}$)) the corresponding left and right traces of $\hat{p}$ (resp. u) at x_i; here a_i also stands for a viscosity distribution, and each choice of a_i corresponds to a different scheme. Furthermore, as we shall see in Sect. 4, for appropriate choices of this parameter we can recover many standard finite volume schemes. Of course, the choice of a_i has a strong impact on the accuracy and the stability of the scheme.

Note that the condition (7) reduces to

$$\sum_i \left(\hat{p}(x_i^+) - \hat{p}(x_i^-)\right)\mu = 0 \quad \forall\,\mu \in \overline{M}_2, \qquad \text{(the last equation of (4))}$$

in the case where $a_i = 0$ or if the function u is continuous at nodes x_i.

Henceforward, we consider that x_i and x_{i+1} are the upstream and the downstream of I_i respectively. Then we define our dual mixed and hybrid finite element method (DMH2) for (5) as follows.

For each $n = 0, 1, \ldots, M-1$, find $(u^{n+1}, \hat{\lambda}) \in M_1 \times \overline{M}_2$, and $\hat{p} \in X$ such that:

$$\begin{cases} \displaystyle\sum_i \int_{I_i} \frac{u^{n+1} - u^n}{\Delta t} v\,dx + \sum_i \int_{I_i} \frac{d\hat{p}}{dx} v\,dx = 0 \quad \forall\, v \in M_1, \\ \displaystyle\sum_i \int_{I_i} \hat{p}\,q\,dx = \sum_i \int_{I_i} \overline{f}(u)\,q\,dx + \sum_i \left(\hat{\lambda}_{i+1}\,q(x_{i+1}^-) \right. \\ \qquad\qquad\qquad \left. - \hat{\lambda}_i q(x_i^+)\right) \quad \forall\, q \in X, \\ \displaystyle\sum_i \left(\hat{p}(x_i^+) - \hat{p}(x_i^-)\right)\mu = \sum_i a_i(\overline{u}_i^+ - \overline{u}_i^-)\,\mu \quad \forall\,\mu \in \overline{M}_2, \end{cases} \tag{8}$$

for $0 \leq i \leq N$. In addition, Eq. (7) can be regarded as the Rankine-Hugoniot jump condition at x_i.

3 Spatial discretization

In this section, we describe how to derive discrete formulations of the method suggested above.

3.1 Scheme 1

Since we seek a solution u in $L^2(I)$, p and $\hat{p}$ in X and $(\lambda, \hat{\lambda})$ in $\overline{M}_2^2$, the choice of functional discrete spaces is $M_{1h} \subset M_1$, $X_h \subset X$ and $M_{2h} \subset \overline{M}_2$, with the spaces defined as

$$
\begin{aligned}
M_{1h} &= \left\{ v_h \in L^2(I);\, v_h|_{(x_i,x_{i+1})} \in \mathbb{P}_0(I_i),\ 0 \le i \le N \right\},\\
X_h &= \left\{ q_h \in X;\, q_h|_{(x_i,x_{i+1})} \in \mathbb{P}_1(I_i),\ 0 \le i \le N \right\},\\
M_{2h} &= \left\{ \mu : \{x_0, x_1, \dots, x_{N+1}\} \to \mathbb{R} \right\},
\end{aligned}
$$

where $\mathbb{P}_k(I_i)$ denotes the space of all polynomials of degree $k\big(k \in \{0, 1\}\big)$ over the interval I_i, and M_{2h} is the global space for the interface unknowns, where functions are defined only at the nodes $\left\{x_i\right\}_{i=0}^{N+1}$.

Using basic algebraic manipulations, we define the contribution of each element within the fully discrete variational formulation:

$$
\begin{cases}
\displaystyle\int_{I_i} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h\, dx - \nu \int_{I_i} \frac{dp_h}{dx} v_h\, dx + \int_{I_i} \frac{d\hat{p}_h}{dx} v_h\, dx = 0 \quad \forall\, v_h \in M_{1h},\\
\displaystyle\int_{I_i} p_h q_h dx = -\int_{I_i} \overline{u}_h \frac{dq_h}{dx}\, dx + \big(\lambda_{i+1} q_h(x_{i+1}^-) - \lambda_i q_h(x_i^+)\big) \quad \forall\, q_h \in X_h,\\
\displaystyle\int_{I_i} \hat{p}_h\, q_h\, dx = \int_{I_i} \overline{f}(u_h)\, q_h\, dx + \big(\hat{\lambda}_{i+1}\, q_h(x_{i+1}^-) - \hat{\lambda}_i q_h(x_i^+)\big) \quad \forall\, q_h \in X_h,
\end{cases}
\tag{9}
$$

for $0 \le i \le N$ and with continuity equations of diffusion and convective fluxes. We have then the following expression of p_h in I_i:

$$
p_h(x) = \frac{\lambda_{i+1} - \lambda_i}{h_i} + 6\frac{(\lambda_{i+1} - 2\overline{u}_{i+1/2} + \lambda_i)}{h_i^2}\big(x - x_{i+\frac{1}{2}}\big),
$$

and the expression of the numerical convective flux $\hat{p}_h$ in I_i:

$$
\hat{p}_h(x) = \overline{f}(u_{i+1/2}) + \frac{\hat{\lambda}_{i+1} - \hat{\lambda}_i}{h_i} + 6\frac{\hat{\lambda}_{i+1} + \hat{\lambda}_i}{h_i^2}\big(x - x_{i+\frac{1}{2}}\big),
$$

where $u_{i+1/2} = u_h(x_{i+1/2})$ and $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$.

Thanks to the expressions of diffusion p_h and convective $\hat{p}_h$ fluxes and continuity equations of p_h and of $\hat{p}_h$, we obtain the scheme:

$$\begin{cases} \dfrac{u_{i+1/2}^{n+1} - u_{i+1/2}^{n}}{\Delta t} - 6\nu \dfrac{\lambda_{i+1} - 2\overline{u}_{i+1/2} + \lambda_i}{h_i^2} + 6\dfrac{(\hat{\lambda}_{i+1} + \hat{\lambda}_i)}{h_i^2} = 0, \\ \text{and the continuity equations of } p_h \text{ and } \hat{p}_h \text{ at node } x_i \text{ are given by} \\ \dfrac{2}{h_{i-1}}\lambda_{i-1} + \left(\dfrac{4}{h_{i-1}} + \dfrac{4}{h_i}\right)\lambda_i + \dfrac{2}{h_i}\lambda_{i+1} = \dfrac{6}{h_{i-1}}\overline{u}_{i-1/2} + \dfrac{6}{h_i}\overline{u}_{i+1/2}, \\ \dfrac{2}{h_{i-1}}\hat{\lambda}_{i-1} + \left(\dfrac{4}{h_{i-1}} + \dfrac{4}{h_i}\right)\hat{\lambda}_i + \dfrac{2}{h_i}\hat{\lambda}_{i+1} = \overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}) \\ \text{for each} \quad n = 0, 1, \dots, M-1, \quad \text{and} \quad 0 \le i \le N. \end{cases}$$

The last two equations express continuity of the fluxes p_h and $\hat{p}_h$ at the interfaces of the cells.

Some remarks are in order.

- Note that since the solution is completely discontinuous, we can eliminate it element by element.
- The present scheme is an implicit, centered and second-order scheme.
- The consistency and conservativity of the scheme are evident.

Remark 1 If we want to obtain the upwind scheme, we have to impose the continuity of the convective flux only on the upstream boundary x_i of each interval I_i (we always assume that we have a right-going wave). For this purpose we use

$$\begin{aligned} c\big((\hat{p}_h, \hat{\lambda}); q_h\big) := & \int_{I_i} \hat{p}_h \, q_h \, dx \\ & - \int_{I_i} \overline{f}(u_h) \, q_h \, dx + \hat{\lambda}_i q_h(x_i^+) = 0 \quad \forall q_h \in X_h. \end{aligned}$$

As above, we get the following upwind scheme:

- equation of conservation of the system (9):

$$\frac{u_{i+1/2}^{n+1} - u_{i+1/2}^{n}}{\Delta t} - 6\nu \frac{\lambda_{i+1} - 2\overline{u}_{i+1/2} + \lambda_i}{h_i^2} + 6\frac{\hat{\lambda}_i}{h_i^2} = 0;$$

- continuity equation of the convective flux at x_i:

$$\frac{2}{h_{i-1}}\hat{\lambda}_{i-1} + \frac{4}{h_i}\hat{\lambda}_i = \overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}).$$

It is evident that the continuity equation of diffusion flux stays unchanged. Note also that the approximation of $\hat{p}$ depends only on $\hat{\lambda}_i$ (we have dropped the parameter $\hat{\lambda}_{i+1}$, insuring the continuity of the convective flux at the downstream boundary x_{i+1} of I_i). For a generalization of this formulation one can see Serghini Mounim [64].

3.2 Scheme 2

In order to construct the approximation of our DMH2 formulation, we keep the above choice of discrete functional spaces M_{1h}, M_{2h} and X_h. Starting from (8), and after a few basic calculations, we obtain the following scheme:

$$\begin{cases} \dfrac{u_{i+1/2}^{n+1} - u_{i+1/2}^{n}}{\Delta t} + 6\dfrac{(\hat{\lambda}_{i+1} + \hat{\lambda}_i)}{h_i^2} = 0, \\ \dfrac{2}{h_{i-1}}\hat{\lambda}_{i-1} + \left(\dfrac{4}{h_{i-1}} + \dfrac{4}{h_i}\right)\hat{\lambda}_i + \dfrac{2}{h_i}\hat{\lambda}_{i+1} = \overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}) \\ \qquad\qquad - a_i(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}) \\ \text{for each} \quad n = 0, 1, \ldots, M-1, \quad \text{and} \quad 0 \leq i \leq N. \end{cases}$$

Observe that the last equation expresses the RH condition of $\hat{p}$ at the boundary x_i, and the expression for $\hat{p}_h$ in I_i is given as

$$\hat{p}_h(x) = \overline{f}(u_{i+1/2}) + \frac{\hat{\lambda}_{i+1} - \hat{\lambda}_i}{h_i} + 6\frac{\hat{\lambda}_{i+1} + \hat{\lambda}_i}{h_i^2}(x - x_{i+\frac{1}{2}}).$$

Note, also, that, since the solution is completely discontinuous, we can eliminate it, element by element and, hence, reduce the number of degrees of freedom.

4 Relationship with finite volume methods

In this section, we show how to obtain finite volume schemes from the mixed and hybrid finite element methods DMH1 and DMH2.

4.1 DMH1

To this end, we use the trapezoidal quadrature formula to approximate the terms $\displaystyle\int_{I_i} l_j l_k \, dx,\ j, k \in \{i, i+1\}$ (coefficients of the local mass matrix, l_i

are the standard hat functions). This beneficially uncouples the equations as we shall see. We first recall the equations of the numerical scheme:

$$\begin{cases} \displaystyle\int_{I_i} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h \, dx - \nu \int_{I_i} \frac{dp_h}{dx} v_h \, dx + \int_{I_i} \frac{d\hat{p}_h}{dx} v_h \, dx = 0 \quad \forall v_h \in M_{1h}, \\ \displaystyle\int_{I_i} p_h q_h dx = -\int_{I_i} \overline{u}_h \frac{dq_h}{dx} \, dx + \big(\lambda_{i+1} q_h(x_{i+1}^-) - \lambda_i q_h(x_i^+)\big) \quad \forall q_h \in X_h, \\ \displaystyle\int_{I_i} \hat{p}_h \, q_h \, dx = \int_{I_i} \overline{f}(u_h) \, q_h \, dx + \big(\hat{\lambda}_{i+1} \, q_h(x_{i+1}^-) - \hat{\lambda}_i q_h(x_i^+)\big) \quad \forall q_h \in X_h. \end{cases}$$

To these equations we have to add the continuity conditions of fluxes p_h and $\hat{p}_h$. Using the trapezoidal quadrature formula and some algebra, we obtain:

$$\begin{cases} \displaystyle\frac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{\Delta t} - 2\nu \frac{\lambda_{i+1} - 2\overline{u}_{i+1/2} + \lambda_i}{h_i^2} + 2\frac{(\hat{\lambda}_{i+1} + \hat{\lambda}_i)}{h_i^2} = 0, \\ \text{and continuity of } p_h \text{ and } \hat{p}_h \text{ at the node } x_i \text{ is given by} \\ \displaystyle\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)\lambda_i = \frac{1}{h_{i-1}} \overline{u}_{i-1/2} + \frac{1}{h_i} \overline{u}_{i+1/2}, \\ \displaystyle\left(\frac{2}{h_{i-1}} + \frac{2}{h_i}\right)\hat{\lambda}_i = \overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}), \\ 0 \le i \le N. \end{cases} \tag{10}$$

In particular, the desired cell average numerical convective flux at the interface cells of I_{i-} and I_i is obtained as $\hat{\alpha}_i = \frac{h_{i-1}\overline{f}(u_{i-1/2}) + h_i \overline{f}(u_{i+1/2})}{h_{i-1}+h_i}$. We observe that this scheme does not contain any upwinding or artificial viscosity. Indeed, we have the following remark.

Remark 2 Due to the elimination of the trace of u, and of the $\hat{\lambda}_i$, at the boundaries of I_i, the finite volume scheme obtained from (10) is:

$$\begin{cases} \displaystyle h_i \frac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{\Delta t} - \nu\left(\frac{\overline{u}_{i+3/2} - \overline{u}_{i+1/2}}{h_{i+\frac{1}{2}}} - \frac{\overline{u}_{i+1/2} - \overline{u}_{i-1/2}}{h_{i-\frac{1}{2}}}\right) \\ \displaystyle + \frac{h_{i+1}}{2} \frac{\overline{f}(u_{i+3/2}) - \overline{f}(u_{i+1/2})}{h_{i+\frac{1}{2}}} + \frac{h_i}{2} \frac{\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2})}{h_{i-\frac{1}{2}}} = 0 \end{cases}$$

for $0 \le i \le N$, and where $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$, $h_{i-\frac{1}{2}} = \frac{1}{2}(h_{i-1} + h_i)$, and $u_{i+1/2}$ is the unknown in interval I_i.

Remark 3 If we consider a spatial mesh with a constant step size, i.e., $x_{i+1} - x_i = h$, then the continuity equations for the diffusion flux become $\lambda_i = \frac{1}{2}(\overline{u}_{i-1/2} + \overline{u}_{i+1/2})$, and for the convective flux $\hat{\lambda}_i = \frac{h}{4}(\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}))$. First, observe that the trace of u between the neighboring cells I_{i-1} and I_i is the average value of u on adjacent sides. Second, one should note that the Lagrange multiplier introduced to relax the continuity of $\hat{p}_h$ across interelement boundaries can be rewritten as $\hat{\lambda}_i \simeq \frac{h^2}{4}\frac{\partial \overline{f}(u)}{\partial x}|_{x_i}$.

4.2 DMH2

Many finite volume schemes can be derived from the dual mixed and hybrid finite element method 2 by using the same techniques as in the previous section. First, we recall the equations which allow us to exhibit the expression of the numerical scheme:

$$\begin{cases} \displaystyle\int_{I_i} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h\,dx + \int_{I_i} \frac{d\hat{p}_h}{dx} v_h\,dx = 0 \quad \forall v_h \in M_{1h}, \\ \displaystyle\int_{I_i} \hat{p}_h\,q_h\,dx = \int_{I_i} \overline{f}(u_h)\,q_h\,dx + \left(\hat{\lambda}_{i+1}\,q_h(x_{i+1}^-) - \hat{\lambda}_i q_h(x_i^+)\right) \quad \forall q_h \in X_h, \end{cases}$$

for $0 \le i \le N$. To these equations are added the jump conditions (RH). If we use the trapezoidal quadrature formula, we get:

$$\begin{cases} \dfrac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{\Delta t} + 2\dfrac{(\hat{\lambda}_{i+1} + \hat{\lambda}_i)}{h_i^2} = 0, \\ \left(\dfrac{2}{h_{i-1}} + \dfrac{2}{h_i}\right)\hat{\lambda}_i = \overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2}) - a_i(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}), \\ 0 \le i \le N. \end{cases}$$

In particular, different convective fluxes for the left and right cells at the interface x_i are obtained:

$$\hat{\alpha}_i^- = \frac{h_{i-1}\overline{f}(u_{i-1/2}) + h_i\overline{f}(u_{i+1/2})}{h_{i-1} + h_i} - a_i\frac{h_{i-1}}{h_{i-1} + h_i}(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}), \tag{11}$$

and

$$\hat{\alpha}_i^+ = \frac{h_{i-1}\overline{f}(u_{i-1/2}) + h_i\overline{f}(u_{i+1/2})}{h_{i-1} + h_i} + a_i\frac{h_i}{h_{i-1} + h_i}(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}), \tag{12}$$

for $0 \le i \le N$. It is important to note that the numerical fluxes used in the FV methods at x_i are our upstream fluxes $F_i = \alpha_i^-$ (we always assume that $a_i \ge 0$, i.e., x_i is the upstream boundary of I_i). Then, we can generalize the above formulation by taking an approach inspired by the upwind method (see [59]). It consists in considering the values of the flux $\hat{p}_h$ at x_i as a convex combination of the upstream fluxes $\hat{p}_h(x_i^-)$ and the downstream fluxes $\hat{p}_h(x_i^+)$ $\big(\hat{p}_h(x_i^\pm) = \hat{\alpha}_i^\pm\big)$:

$$\hat{p}_h(x_i) = \begin{cases} \dfrac{1+\beta}{2}\hat{p}_h(x_i^-) + \dfrac{1-\beta}{2}\hat{p}_h(x_i^+) & \text{if } a_i > 0, \\ \dfrac{1-\beta}{2}\hat{p}_h(x_i^-) + \dfrac{1+\beta}{2}\hat{p}_h(x_i^+) & \text{if } a_i < 0, \end{cases} \tag{13}$$

where $\beta \in [0, 1]$ is the upwinding parameter, with $a_i > 0$ (resp. < 0) if x_i is the upstream (resp. downstream) boundary of I_i. Consequently, by means of (13), we also obtain the local conservativity of the scheme. Then, we replace, in each I_i, $\int_{I_i} \frac{d\hat{p}_h}{dx} v_h\, dx$ by $(\frac{1+\beta}{2}\hat{p}_h(x_{i+1}^-) + \frac{1-\beta}{2}\hat{p}_h(x_{i+1}^+))v_h|_{x_{i+1}^-} - (\frac{1+\beta}{2}\hat{p}_h(x_i^-) + \frac{1-\beta}{2}\hat{p}_h(x_i^+))v_h|_{x_i^+}$ for all $v_h \in M_{1h}$.

Remark 4 By again denoting the unknown in I_i by $u_{i+1/2}$, and using the same notation as above, the general finite volume scheme obtained from the association of (11), (12) and (13) with $\beta = 1$, is:

$$\begin{cases} h_i \dfrac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{\Delta t} \\ + \dfrac{h_{i+1}}{2} \dfrac{\overline{f}(u_{i+3/2}) - \overline{f}(u_{i+1/2})}{h_{i+\frac{1}{2}}} + \dfrac{h_i}{2} \dfrac{\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2})}{h_{i-\frac{1}{2}}} \\ - a_{i+1} \dfrac{h_i}{2} \dfrac{\overline{u}_{i+3/2} - \overline{u}_{i+1/2}}{h_{i+\frac{1}{2}}} + a_i \dfrac{h_{i-1}}{2} \dfrac{\overline{u}_{i+1/2} - \overline{u}_{i-1/2}}{h_{i-\frac{1}{2}}} = 0, \end{cases}$$

whereas the choice of $\beta = 0$ (as well as $a_i = 0$) gives the centered scheme.

Remark 5 If we consider a spatial mesh with a constant step size, i.e., $h = x_{i+1} - x_i$, thanks to the jump condition we can obtain

$$\hat{\lambda}_i = \frac{h}{4}\Big[\big(\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2})\big) - a_i(\overline{u}_{i+1/2} - \overline{u}_{i-1/2})\Big].$$

Hence, we can state that the Rankine-Hugoniot jump condition at x_i is recovered in the expression of the Lagrange multiplier.

Note that the elimination of the Lagrange parameter $\hat{\lambda}$ implies the lack of smoothing in the resulting finite volume scheme.

Many choices of speed a_i are possible. We can take

$$a_i = \left| f'\left(\frac{\overline{u}_{i-1/2} + \overline{u}_{i+1/2}}{2}\right) \right|$$

as a local speed at x_i, which suggests the following numerical flux for the Burgers equation

$$\hat{\alpha}_i^- = \frac{h_{i-1}\overline{u}_{i-1/2}^2 + h_i\overline{u}_{i+1/2}^2}{2(h_{i-1}+h_i)} - |\overline{u}_i| \frac{h_{i-1}}{h_{i-1}+h_i}(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}),$$

where $\overline{u}_i = \frac{1}{2}(\overline{u}_{i-1/2} + \overline{u}_{i+1/2})$.

We can also take $a_i = \max(|f'(v)|)$ over all v between $\overline{u}_{i-1/2}$ and $\overline{u}_{i+1/2}$, and in the special case of convex flux f, this is further simplified: $a_i = \max(|f'(\overline{u}_{i-1/2})|, |f'(\overline{u}_{i+1/2})|)$. This last choice results in Rusanov's method which is often called the local Lax-Friedrichs method (LLF). Another related method is that of Murman, where a_i is given at x_i by

$$a_i = \left| \frac{\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2})}{\overline{u}_{i+1/2} - \overline{u}_{i-1/2}} \right|.$$

Unlike the LLF scheme, solutions generated with this method may fail to satisfy the entropy condition.

Many numerical fluxes are written in the form (11). See, for example, Le Roux [47] or LeVeque [49,50]. Without loss of generality, we concentrate only on a uniform spacial mesh, i.e., $h = x_{i+1} - x_i$ (constant step size). Below we show the relationships between the various choices of the local speed a_i and the standard finite volume schemes.

- **The second-order central differencing scheme:**

$$a_i = 0.$$

- **The first-order upwind scheme:**

$$a_i = \begin{cases} \left| \dfrac{\overline{f}(u_{i+1/2}) - \overline{f}(u_{i-1/2})}{\overline{u}_{i+1/2} - \overline{u}_{i-1/2}} \right| & \text{if } \overline{u}_{i+1/2} - \overline{u}_{i-1/2} \neq 0, \\[2ex] \left| \dfrac{\partial \overline{f}(u)}{\partial u} \right| & \text{otherwise.} \end{cases}$$

- **Lax-Friedrichs scheme:**

$$a_i = \max\big(|\overline{f}'(v)|\big),$$

where the maximum is taken over the whole region in which $\overline{u}_{i-1/2}, \overline{u}_{i+1/2}$ varies, i.e., in $\big[\inf u_0(x), \sup u_0(x)\big]$, where $u_0(x)$ is again the initial function. In the linear case $\big(f(u) = au\big)$, we recover the Lax-Friedrichs scheme by setting $a_i = \frac{h}{\Delta t}$.

- **Lax-Wendroff scheme:**

$$a_i = \frac{\Delta t}{h}\big(f'(\xi_i)\big)^2,$$

where

$$\begin{cases} \xi_i \in I(\overline{u}_{i-1/2}, \overline{u}_{i+1/2}) \quad \text{and satisfies} \\ f(\overline{u}_{i+1/2}) - f(\overline{u}_{i-1/2}) = (\overline{u}_{i+1/2} - \overline{u}_{i-1/2})\dfrac{\partial f}{\partial u}(\xi_i). \end{cases} \tag{14}$$

- **Godunov scheme:**

$$a_i = (1-\gamma)f'(\xi_{i+1/4}) + \gamma f'(\xi_{i-1/4}),$$

where $\gamma \in [0, 1]$ is defined by $\overline{u}_{i+1/2} = (1-\gamma)\overline{u}_i + \gamma\overline{u}_{i+1}$, and

$$\begin{cases} \xi_{i+1/4} \in I(\overline{u}_i, \overline{u}_{i+1/2}) \quad \text{and satisfies} \\ f(\overline{u}_{i+1/2}) - f(\overline{u}_i) = (\overline{u}_{i+1/2} - \overline{u}_i)\dfrac{\partial f}{\partial u}(\xi_{i+1/4}). \end{cases}$$

The value $\xi_{i-1/4} \in I(\overline{u}_{i-1/2}, \overline{u}_i)$ is defined in an analogous way.

- **Roe scheme:**

$$a_i = |A_{R_{|i}}|,$$

where $A_{R_{|i}} = A_R(\overline{u}_{i-1/2}, \overline{u}_{i+1/2})$ is the Roe average, associated to the Jacobian A of f.

- **Engquist-Osher scheme:**
The Engquist-Osher method takes the opposite approach to Godunov's method in which we always use the shock-wave solution to each Riemann problem, and always assumes the solution is a "rarefaction wave". This can be accomplished by setting

$$\sum_i \big(\hat{p}_h(x_i^+) - \hat{p}_h(x_i^-)\big)\,\mu = \sum_i -\bigg(\int_{\overline{u}_{i-1/2}}^{\overline{u}_{i+1/2}} \big|f'(v)\big|dv\bigg)\,\mu \quad \forall \mu \in \overline{M}_{2h}.$$

For the above schemes, see Le Roux [47] or LeVeque [49,50] and references. In some forthcoming cases, the trial space $(\widetilde{M}_{1h})$ is chosen to be different from the test space (M_{1h}), i.e., the Petrov-Galerkin DMH formulation type.

- **Nessyahu-Tadmor (NT) scheme [55]:**
  Assume that we replace piecewise cell values by piecewise linear MUSCL-type interpolants $\big(\widetilde{M}_{1h} = \{u_h \in L^2(I); u_h|_{(x_i,x_{i+1})} \in \mathbb{P}_1(I_i), 0 \le i \le N\}\big)$. Then at interface x_i, we have left and right values from the two linear approximations in each of the neighboring cells. Denote these values by

$$\overline{u}_i^- = \overline{u}_{i-1/2} + \frac{h}{2}\sigma_{i-1/2} \quad \text{and} \quad \overline{u}_i^+ = \overline{u}_{i+1/2} - \frac{h}{2}\sigma_{i+1/2},$$

  where $\sigma_{i-1/2}$ and $\sigma_{i+1/2}$ are the left and right slopes at cell boundary x_i.
  If we replace in the second term of the general form (11), $\overline{u}_{i-1/2}$ and $\overline{u}_{i+1/2}$ respectively by $\overline{u}_i^-$ and $\overline{u}_i^+$ with $a_i = \frac{h}{\Delta t}$, this results in a second-order, non-oscillatory central scheme introduced by Nessyahu and Tadmor in [55]:

$$\hat{\alpha}_i^- = \frac{\overline{f}(u_{i-1/2}) + \overline{f}(u_{i+1/2})}{2} - \frac{a_i}{2}(\overline{u}_i^+ - \overline{u}_i^-), \tag{15}$$

  or equivalently

$$\hat{\alpha}_i^- = \frac{\overline{f}(u_{i-1/2}) + \overline{f}(u_{i+1/2})}{2} - \frac{a_i}{2}(\overline{u}_{i+1/2} - \overline{u}_{i-1/2}) + \frac{ha_i}{4}(\sigma_{i+1/2} + \sigma_{i-1/2}),$$

  where the reconstruction is based on staggered grids. The numerical flux is approximated by the second-order midpoint quadrature rule

$$\overline{f}(u_{i+1/2}) \simeq f\big(u_{i+1/2}^{n+1/2}\big),$$

  and the pointwise values at the half-time steps are evaluated by a Taylor expansion (for details, see [55]).
- **The β scheme [8]:**
  Here the trial space is M_{1h}, and the left and right control volume boundary states $\overline{u}_i^-$ and $\overline{u}_i^+$ that are involved in Eq. (15), are computed by using piecewise linear interpolation formulas:

$$\overline{u}_i^- = \overline{u}_{i-1/2} + \frac{h}{2}\big((1-\beta)\,\widetilde{\sigma}_{i+1/2} + \beta\,\widetilde{\sigma}_{i-1/2}\big), \tag{16}$$

  and

$$\overline{u}_i^+ = \overline{u}_{i+1/2} - \frac{h}{2}\big((1-\beta)\,\widetilde{\sigma}_{i+1/2} + \beta\,\widetilde{\sigma}_{i+3/2}\big), \tag{17}$$

  where β is an upwinding parameter that controls the combination of fully upwind and centered slopes, and $\widetilde{\sigma}_{i+1/2} = \frac{u_{i+1/2}-u_{i-1/2}}{h}$ denotes the left slope at cell boundary, $x_{i+1/2}$.
  The β scheme results from the combination of (15), (16) and (17), with the speed at x_i computed as $a_i = \delta\big|f'\big(\frac{\overline{u}_{i-1/2}+\overline{u}_{i+1/2}}{2}\big)\big|$. Note that the role of the parameter δ is to minimize the dissipative error; for more details see [8].

- **Kurganov-Levy scheme [42]:**
  In a similar way, we can consider u as a piecewise parabolic function here (instead of the piecewise linear one, employed in the NT scheme), and we define P_i in each interval I_i, by $P_i(x) = A_i + B_i(x - x_{i+1/2}) + \frac{1}{2}C_i(x - x_{i+1/2})^2$, where the values of A_i, B_i and C_i are
  $A_i = u_{i+1/2} - \frac{w_c}{12}(u_{i+3/2} - 2u_{i+1/2} + u_{i-1/2}),$
  $B_i = \frac{1}{h}\Big[w_r(u_{i+3/2}-u_{i+1/2})+w_c\frac{u_{i+3/2} - u_{i-1/2}}{2}+w_l(u_{i+1/2}-u_{i-1/2})\Big],$
  $C_i = w_c\frac{u_{i-1/2} - 2u_{i+1/2} + u_{i+3/2}}{h^2},$
  where the weights w_l, w_c and w_r ($w_i \geq 0 \forall i \in \{l, c, r\}$, and $\sum_i w_i = 1$), are defined in [42].
  The left and right intermediate values u_i^- and u_i^+ of $u(x, t^n)$ at x_i are obtained as $u_i^- := P_{i-1}(x_i^-)$ and $u_i^+ := P_i(x_i^+)$. Therefore, a similar approach leads to the third-order scheme

$$\hat{\alpha}_i^- = \frac{\overline{f}(u_i^-) + \overline{f}(u_i^+)}{2} - \frac{a_i}{2}(\overline{u}_i^+ - \overline{u}_i^-),$$

  which is based on the reconstruction CWENO (the expression of the speed a_i at node x_i is given in [42]).

We note that the class of schemes (11) has been studied in Le Roux [45, 46], where they are derived from finite difference methods. In particular, we can find there the proof of the following result.

Theorem 6 *Let* $M = \sup_{u,x,t}\left\{\left|\frac{\partial f}{\partial u}(u, x, t)\right|\right\}$ *be the uniform Lipschitz constant of f with respect to the variable u, and let the initial data u_0 be in the space $L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$. If the stability condition (CFL) $M\frac{\Delta t}{h} \leq 1$ is satisfied, and if for all $h > 0$ the choice of the coefficients a_i is such that*

$$\forall i \in Z, \quad \forall n \leq N, \quad \left|f'(\xi_i)\right| \leq a_i \leq \frac{h}{\Delta t},$$

is defined by (14), then the family of approached solutions $\{u_h\}_{h>0}$ contains a subsequence in $L^1_{loc}(\mathbb{R}\times]0, T[)$ that converges to the weak solution u of (5) where $u \in L^\infty(\mathbb{R}\times]0, T[) \cap BV_{loc}(\mathbb{R}\times]0, T[)$.

Note that the scheme with a local propagation speed $a_i = |f'(\xi_i)|$ (see the above theorem) can sometimes give weak solutions that fail to satisfy the entropy condition (see [46]).

5 Stabilization of scheme 1 with the flux-limiter method

A rigorous theoretical demonstration of TVD properties of the scheme, i.e.,

$$\sum_{i=-\infty}^{\infty} \left|u_{i+1}^{n+1} - u_i^{n+1}\right| \leq \sum_{i=-\infty}^{\infty} \left|u_{i+1}^{n} - u_i^{n}\right|,$$

once we have applied the flux-limiter method, is delicate because the scheme is vectorial and implicit. Nonetheless, we can use the following strategy to stabilize the scheme.

We have seen the fact that imposing continuity to convective flux yields a conservative central scheme and second-order spatial accuracy, but this also where problems arise. Since continuity was imposed with a Lagrange multiplier $\hat{\lambda}$, it is obvious that the flux must be controlled when necessary, especially if we do not want the scheme to amplify extreme points of the solution or to create new ones. What we now try to show is that the standard flux limiter schemes correspond to a relaxation of the continuity properties of the flux at interfaces, by imposing a constraint on the Lagrange multiplier. To simplify recall Remark 3, where we observed that $\hat{\lambda}_i = \frac{ah}{4}(u_{i+1/2} - u_{i-1/2})$ if we assume $f(u) = au$, $a > 0$ (obviously, a similar method can be defined when $a < 0$).

This is so that we can write the numerical convective flux $\hat{p}$, at the node x_{i+1} considered as downstream of the interval I_i, as (as in Sect. 4, we use the trapezoidal quadrature formula to diagonalize the local mass matrix),

$$\hat{p}_{i+1} = f(u_{i+1/2}) + g_{i+1},$$

in which $g_{i+1} = \frac{2}{h}\hat{\lambda}_{i+1}$ is interpreted as an anti-diffusive flux, insuring continuity of the convective flux at downstream. We have seen as well that it approximates

$$\frac{h}{2}\frac{\partial f(u)}{\partial x}\bigg|_{x_{i+1}},$$

and thus, from

$$\int_{I_i} \frac{d\hat{p}_h}{dx}\,dx = 2\frac{(\hat{\lambda}_{i+1} + \hat{\lambda}_i)}{h_i},$$

it cancels the first-order error. In order to achieve a second-order accurate scheme which satisfies the TVD condition, one needs to correct the flux. In fact, it is necessary to limit the anti-diffusive flux. For this purpose, we define in a classical way

$$\tilde{g}_{i+1} = B(g_{i+1}, g_i)$$

where B is a function that satisfies the condition

$$B(r, r) = r.$$

A standard choice is the minmod function

$$B(r,s) = \begin{cases} 0 & \text{if } r \quad\quad\quad\quad\quad\quad\;\; \text{and } s \text{ have opposite signs,} \\ r & \text{if } |r| \le |s| \text{ and } r \text{ and } s \text{ have the same sign,} \\ s & \text{if } |s| \le |r| \text{ and } r \text{ and } s \text{ have the same sign.} \end{cases}$$

Alternative flux limiter functions were studied by Roe [61] and Sweby [67] using the ratio of consecutive gradients $\hat{\lambda}$,

$$\theta_i = \frac{u_{i+1/2} - u_{i-1/2}}{u_{i+3/2} - u_{i+1/2}} = \frac{\hat{\lambda}_i}{\hat{\lambda}_{i+1}}.$$

Then one can set

$$B(g_{i+1}, g_i) = \phi(\theta_i) g_{i+1},$$

where the function ϕ satisfies the symmetry condition

$$\phi(\theta) = \theta \phi\left(\frac{1}{\theta}\right)$$

and

$$\phi(1) = 1.$$

If $\hat{\lambda}_i \hat{\lambda}_{i+1} \le 0$ or $\theta_i \le 0$, we are in the presence of spurious oscillations near $u_{i+1/2}$, and one takes $\tilde{g}_{i+1} = 0$ or, if we want less diffusion, we set $B(g_{i+1}, g_i) = (1-\beta) g_i + \beta g_{i+1}$ ($0 \le \alpha \le 1$ is the upwinding parameter). For the flux limiter $\Phi_i = \phi(\theta_i)$, we choose the standard ones in the literature (see Sect. 7), and we set $\tilde{g}_{i+1} = \Phi_i g_{i+1}$.

5.1 Hybrid scheme

Along the same lines, we can define a novel scheme in two steps. This hybrid scheme consists in combining two schemes of first- and second-order accuracy. Hence, in smooth regions the high order scheme is used to guarantee the maximum order of accuracy. However, in the presence of large gradients or discontinuities, this reconstruction switches to one of the first-order (the low order flux is favored). Finally, the procedure is as follows. The passage from the solution u^n at the time level $t^n = n\Delta t$ to the solution u^{n+1} at the time level $t^{n+1} = (n+1)\Delta t$, where Δt again designates the time step, is made in two steps.

- First, we apply the upwind scheme. In particular, we obtain $u^{n+1/2}$ and $\hat{\lambda}^I$.
- In the second and final step, we use the centered scheme, with $u^{n+1/2}$ as initial data. Here $\hat{\lambda}^I$ is preferred to $\hat{\lambda}^S$ when necessary (i.e., $\hat{\lambda}_i \hat{\lambda}_{i+1} \le 0$), as explained above, and reciprocally. We denote the final solution by u^{n+1}, where we have used the notation:

– $\hat{\lambda}^I$ and $\hat{\lambda}^S$ are the resulting fluxes.
– $\hat{\lambda}^I$: low order flux with the upwind scheme.
– $\hat{\lambda}^S$: high order flux with the centered scheme.

The fact that the continuity of fluxes has been sacrified implies that the conservativity of finite volume schemes is not as good as what is usually stated. What is conserved is a modified flux, and not the flux itself.

Finally, it is important to observe that the fact of introducing the Lagrange multiplier to relax the continuity of $\hat{p}$ at the cell interfaces, is the source of the following features.

– Relationship with the finite volume method.
– Local conservativity of the scheme.
– Stabilization via the upwind and the flux limiter methods.
– Advantage that no (approximate) Riemann solvers are required.

6 Discussion on DMH 2

1 The extension to a system of equations can be carried out with the aid of Roe's construction [62]. The local speed $a_i(u_{i-1/2}, u_{i+1/2})$ $\big(f(u_{i+1/2}) - f(u_{i-1/2}) = a_i(u_{i+1/2} - u_{i-1/2})\big)$, is replaced by the corresponding matrices $\hat{A}(u_{i-1/2}, u_{i+1/2})$ (the Rankine-Hugoniot conditions are a system of N equations). Here $\hat{A}$ is the matrix which approximates the Jacobian matrix $\partial \underline{f}/\partial u$, and which satisfies

$$\hat{A}(u_{i-1/2}, u_{i+1/2})(u_{i-1/2}, u_{i+1/2}) = \underline{f}(u_{i+1/2}) - \underline{f}(u_{i-1/2}),$$

with $\underline{f}$ being a N-component flux vector. Then, $\hat{A}(u_{i-1/2}, u_{i+1/2})$ is expanded in terms of the eigenvectors of $\hat{A}(u_{i-1/2}, u_{i+1/2})$, and a contribution to the dissipation term is formed by multiplying each eigenvector by a coefficient with a magnitude not less than that of the corresponding eigenvalue. Shortly, we will replace $a_i(u_{i-1/2}, u_{i+1/2})$ by $\Gamma_i|\Lambda_i|\Gamma_i^{-1}$, where $|\hat{A}| = \Gamma|\Lambda|\Gamma^{-1}$; here, Γ denotes the matrix of the right eigenvectors of $\hat{A}$ and Λ the diagonal matrix of the eigenvalues of the system.

2 The extension to a convection-diffusion equation can be performed as follows.

The idea is that the equation

$$\frac{\partial u}{\partial t} + \frac{\partial\big(f(u)\big)}{\partial x} = 0, \tag{18}$$

arises by assuming that the viscous term in the equation

$$\frac{\partial u}{\partial t} + \frac{\partial\big(f(u)\big)}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{19}$$

is small, and then setting it to zero. It is well-known that the weak solution of (18) with the initial condition

$$u(x,0) = \begin{cases} u^L & \text{if} \quad x < 0, \\ u^R & \text{if} \quad x > 0, \end{cases}$$

is the $u(x-st)$ step function whose discontinuity propagates with speed s. This solution is to be regarded as a solution of (18) provided the viscous equation (19) has a traveling-solution $u(x-st)$ with $\lim\limits_{\chi \longrightarrow \pm\infty} u(\chi) = u_{R,L}$, $\lim\limits_{\chi \longrightarrow \pm\infty} u'(\chi) = 0$ ($u((x-st)/\nu)$ converges to $u(x-st)$ as $\nu \to 0$, the viscosity tends to zero), and the shock waves $u(x-st)$ are limits of traveling-wave solutions of (19) as $\nu \to 0$ (the vanishing weak solution). In other words, the solutions to the general initial value problem of the viscous conservation laws converge, in the zero dissipation limit, as ν goes zero, to the solutions of the hyperbolic conservation laws. Further, we recover, when ν is near 0, the Rankine-Hugoniot jump condition $f(u^R) - f(u^L) - s(u^R - u^L) = 0$.

Consider now the convection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial\big(F(u)\big)}{\partial x} = 0 \quad \forall x \in \mathbb{R} \times (0, +\infty),$$

where the general flux consists of convection and diffusion fluxes

$$F(u) = f(u) - \nu \frac{\partial u}{\partial x}.$$

If the trajectory of a shock Σ is determined by $x(t)$, we have at the limit $(\nu \longrightarrow 0)$

$$\big[F(u)\big] = s[u], \quad s = \frac{dx}{dt} \quad \text{is the shock speed,}$$

where $[u] = u\big(x(t)^+, t\big) - u\big(x(t)^-, t\big)$, and $\big[F(u)\big] = F\big(u(x(t)^+, t)\big) - F\big(u(x(t)^-, t)\big)$ stand for the jumps of u and $F(u)$ across the shock Σ.

Moreover, we set $\hat{P}_h = -\nu \frac{\partial u_h}{\partial x} + f(u_h)$ (the general numerical flux), and we have the discrete problem:
find

$$\big(u_h(t), \hat{\lambda}(t)\big) \in M_{1h} \times \overline{M}_{2h}, \text{ and } \hat{P}_h(t) \in X_h$$

such that:

$$\int_I \frac{\partial u_h}{\partial t} v_h \, dx + \sum_{I_i} \int_{I_i} \frac{d\hat{P}_h}{dx} v_h \, dx = 0 \qquad \forall v_h \in M_{1h},$$

$$\int_I \hat{P}_h . \underline{q}_h dx = \sum_{I_i} \int_{I_i} \nu u_h \frac{dq_h}{dx} dx +$$
$$\sum_{I_i} \int_{I_i} f(u_h) q_h \, dx - \sum_i \left(\lambda_{i+1} q_h(x_{i+1}^-) - \lambda_i q_h(x_i^+) \right) \qquad \forall q_h \in X_h,$$

$$\sum_i [\hat{P}_h] \, \mu_h = \sum_i a_i [u_h] \, \mu_h \qquad \forall \mu_h \in \overline{M}_{2h},$$

where again $[P_h] = P_h^+ - P_h^-$, $[u_h] = u_h^+ - u_h^-$, and a_i denotes the approximation of the local speed at x_i.

7 Numerical results

In this section, we use two one-dimensional model problems to test our schemes numerically. The aim here is to show the feasibility of the methods.

Example 7 (Transport equation and propagation of singularities.) We solve the model advection-diffusion equation

$$u_t + u_x = \nu (u_x)_x, \quad 0 \leq x \leq 1,$$

subject to initial data $u(x, 0) = u_0(x)$. Here, we consider the discontinuous characteristic function, $u_0(x) = \chi_0 = \mathbb{I}_{(0.11111, 0.4)}$, with diffusion parameter $\nu = 10^{-16} (\nu \longrightarrow 0)$ and boundary conditions $u(0, t) = u(1, t) = 0.1$. With such a piecewise constant initial condition, we can easily illustrate the dispersive and dissipative character of the many versions of the scheme.

- In Figs. 1 and 2, the physical and numerical coefficients are chosen to be those given in the table below.

Figure	Mesh points	Iterations	CFL
a	100	26	1/3
b	50	26	1/6
c	75	13	1/2

- In Fig. 1, we observe the good resolution of the computed solution. Note, in particular, that for $\alpha = 1$ the scheme is still compressive.

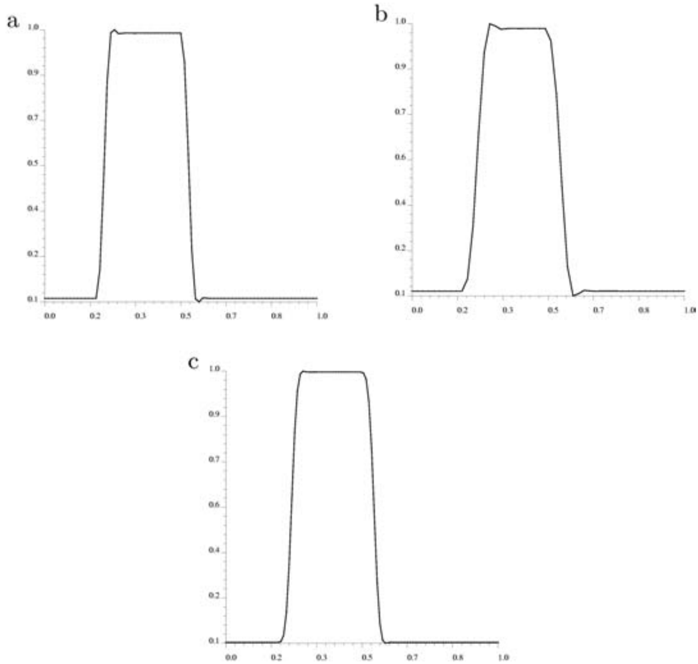

Fig. 1. Upwind scheme

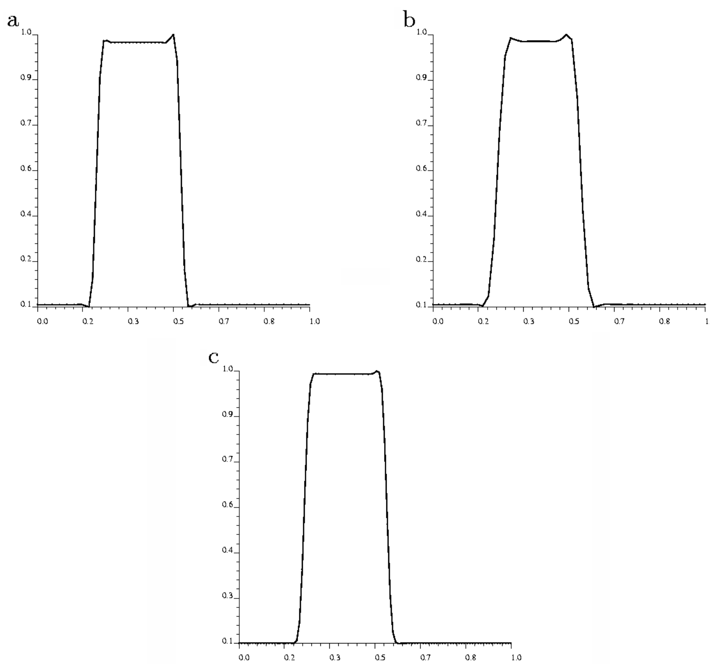

Fig. 2. Hybrid scheme

- The results of the hybrid scheme are presented in Fig. 2, associated with the implicit Euler method. It shows the possibility of using the flux-limiter method as a stabilizer method, and also the efficiency of the scheme to capture the discontinuities and shocks.
- In Figs. $3, 4, \ldots, 10$, the calculations were carried out on a uniform grid of 75 mesh points, with time step size chosen as CFL $=$ 0.5 and 13 iterations. The temporal approximation used is the implicit Euler method combined with standard flux limiters. In the presence of spurious oscillations we use two strategies to modify the "flux" λ (shock or discontinuity detector):

  abandoned continuity: we set $\hat{\lambda}_{i+1} = 0$,

  relaxed continuity: we set $\hat{\lambda}_{i+1} = (1-\beta)\hat{\lambda}_i + \beta\hat{\lambda}_{i+1}$.

  Lastly, when we modify the flux $\hat{\lambda}$, only in presence of oscillations, we refer to this process as local limiter. We can see the performance of the flux-limiter method, which confirms our idea of using it as stabilizer method. The best results are now obtained for the "abandoned continuity" choice, which have no spurious oscillations near the shock or discontinuity.

Example 8 (Expansion of discontinuity and moving shock.) In the following, we want to test the validity of the stabilized schemes (upwinding and flux-limiter methods) on the nonlinear equation. To achieve this, we approximate solutions to the well-known one-dimensional Burgers equations.

$$u_t + (\frac{u^2}{2})_x = \nu((u)_x)_x, \quad 0 \le x \le 1,$$

with the initial condition of $u(x,0) = u_0(x)$, and where

$$\text{discontinuity plus shock wave} = \begin{cases} u_0 = \begin{cases} 1 & \text{if} \quad 0.11111 \le x \le 0.4 \\ 0.1 & \text{otherwise} \end{cases} \\ u(0,t) = 0.1, \quad u(1,t) = 0.1. \end{cases}$$

The exact solution ($\nu \to 0$) develops a rarefaction wave with a moving shock with a velocity of $u_s = \frac{u_L + u_R}{2}$, where u_L and u_R are the speed on both sides of the shock front. For the numerical results, here and below, $\mu = \frac{h}{\Delta t}$ denotes the fixed mesh-ratio in the x-direction. This test case contains both expansion of discontinuity and propagation of shock. We observe that the scheme is able to capture the moving discontinuity with a few transition points, and gives a particularly good representation of the rarefaction wave (in both stages of the developed rarefaction wave), once we take into account the simplicity and efficiency offered by our scheme.

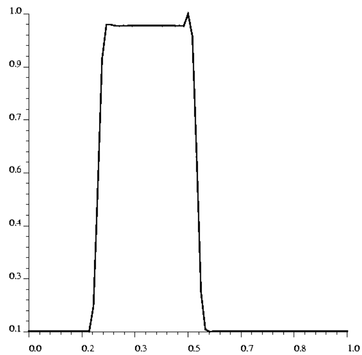

Fig. 3. Flux-limiter: Van Leer relaxed continuity, $\beta = 5/7$

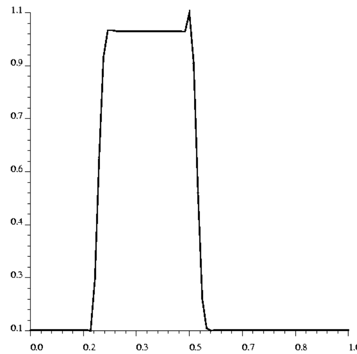

Fig. 4. Flux-limiter: minmod relaxed continuity, $\beta = 5/7$

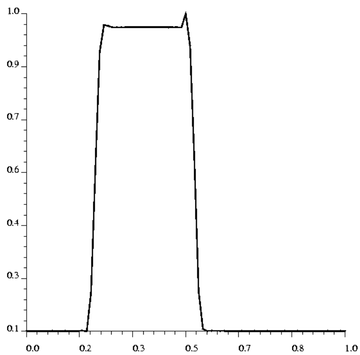

Fig. 5. Flux-limiter: Chakravarty relaxed continuity, $\beta = 5/7$

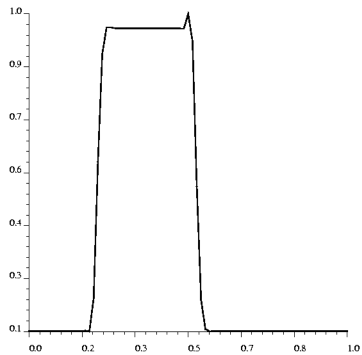

Fig. 6. Flux-limiter: local relaxed continuity, $\beta = 5/7$

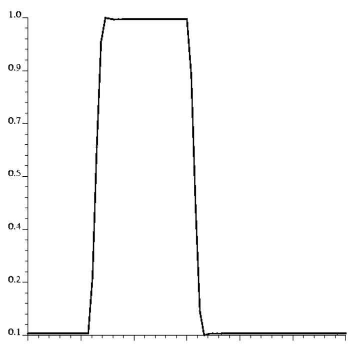

Fig. 7. Flux-limiter: Van Leer abandoned continuity

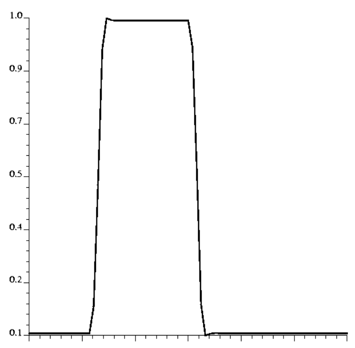

Fig. 8. Flux-limiter: minmod abandoned continuity

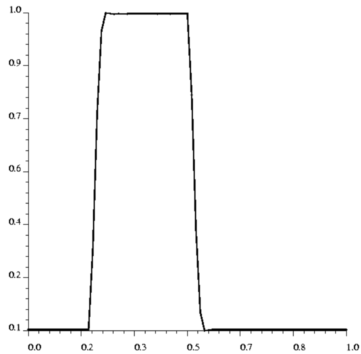

Fig. 9. Flux-limiter: Chakravarty abandoned continuity

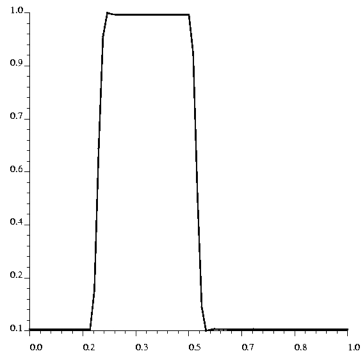

Fig. 10. Flux-limiter: local abandoned continuity

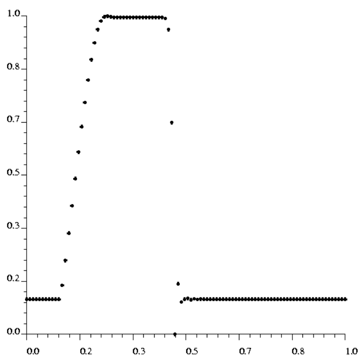

Fig. 11. Upwind scheme, Burgers $\mu = 0.1, h = 0.01$, 100 iterations

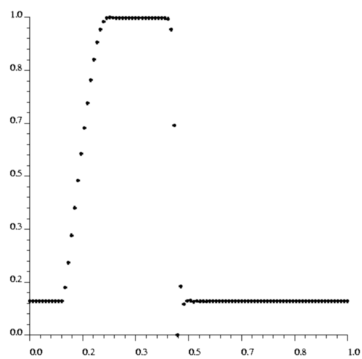

Fig. 12. Upwind scheme, Burgers $\mu = 0.5, h = 0.01$, 20 iterations

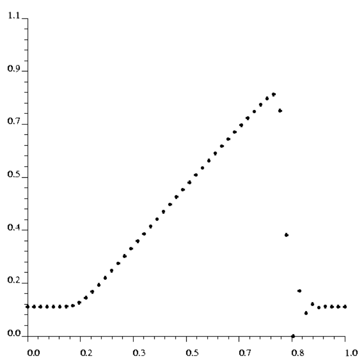

Fig. 13. Upwind scheme, Burgers $\mu = 0.5, h = 0.02$, 75 iterations

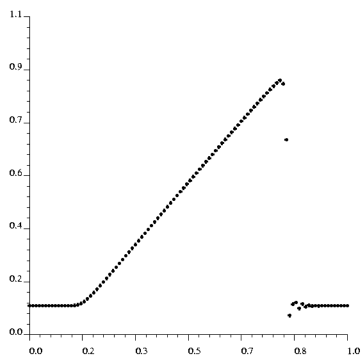

Fig. 14. Upwind scheme, Burgers $\mu = 0.5, h = 0.01$, 150 iterations

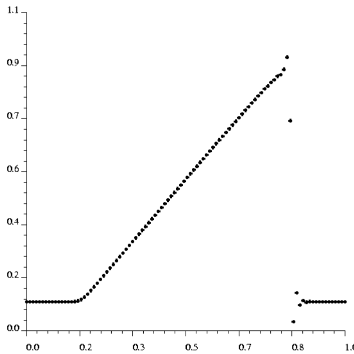

Fig. 15. Flux-limiter Van Leer, abandoned continuity, same parameters as Fig. 14

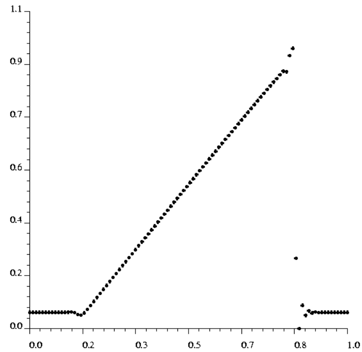

Fig. 16. Flux-limiter local limiter, abandoned continuity, same parameters as Fig. 14

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